JOURNAL OF DIFFERENTIAL EQUATIONS **97**, 71–98 (1992)

Rational Solutions of Nonlinear Evolution Equations, Vertex Operators, and Bispectrality

JORGE P. ZUBELLI*

Department of Mathematics, University of California, Berkeley, California 94720

Received March 20, 1990; revised September 10, 1990

We prove that if $q(x, t_2, ..., t_m)$ and $r(x, t_2, ..., t_m)$ are certain rational solutions of the AKNS hierarchy, then there are eigenfunctions $\psi(x, k)$ of the AKNS/ZS operator

$$\mathbf{L} = \begin{bmatrix} \partial_x & -q \\ r & -\partial_x \end{bmatrix}$$

satisfying a differential equation in the spectral parameter k of the form $B(k, \partial_k)\psi = \Theta(x)\psi$, where $B(\partial_k, k)$ is a matrix differential operator, independent of x, and Θ is a nonconstant function of x. We also discuss the relation between this result and a similar one for the rational solutions of the Schrödinger operator with potentials in the manifold of rational solutions of the KdV hierarchy. © 1992 Academic Press, Inc.

1. INTRODUCTION

One of the most fascinating aspects of the theory of solitons is its connection with a vast number of seemingly unrelated problems. One of these problems was studied by J. J. Duistermaat and F. A. Grünbaum for the Schrödinger operator $-\partial_x^2 + v(x)$ in [9]. Its general form is as follows: Given a differential operator $L(x, \partial_x)$, when do we have a family of solutions $\varphi(x, k)$ of

$$L\varphi = k\varphi \tag{1}$$

that also satisfies a differential equation in the spectral parameter k, of the form

$$B(k, \partial_k)\varphi = \Theta(x)\varphi, \tag{2}$$

* Research supported in part by ONR Contract N00014-84-C-0159 and NSF Grant DMS 87-20007.

0022-0396/92 $3.00

where B is a differential operator in k, independent of x, and $\Theta(x)$ is a nonconstant function of x? In other words, φ is an eigenfunction for a differential operator in x, with spectral parameter k, and also an eigenfunction for a differential operator in k with spectral parameter $\Theta(x)$. This motivates us to call the nonzero common solutions of (1) and (2) *bispectral eigenfunctions*. In this case we shall also say that L has the *bispectral property*. In [9] all the bispectral Schrödinger operators were characterized. The remarkable fact is that a distinguished class of potentials $v(x)$ such that $-\partial_x^2+v$ is bispectral coincides with the rational solutions of the Korteweg–de Vries (KdV) equation. By rational solutions of the KdV we mean the rational functions, decaying at infinity, that remain rational by the flow of the KdV equation.

The main purpose of this paper is to establish the bispectral property of the AKNS/ZS operator

$$\mathbf{L}=\begin{bmatrix}\partial_x & -q\\ r & -\partial_x\end{bmatrix}, \tag{3}$$

for a certain class of rational functions (q, r) that stay rational under the flows of the AKNS hierarchy of nonlinear evolution equations [1, 22, 20, 17]. This hierarchy was introduced in [1] and it includes, under certain specializations, the hierarchy of the modified Korteweg–de Vries equation and the hierarchy of the nonlinear Schrödinger equation. More specifically, we show that if (q, r) is a rational solution of the AKNS hierarchy, satisfying the two conditions described below, then the operator $\mathbf{L}$ in Eq. (3) has the bispectral property. We explicitly exhibit bispectral eigenfunctions, which are obtained using vertex operators. Such operators play an important role in the study of the hierarchies of completely integrable equations and connect this subject with the theory of infinite dimensional Lie algebras [7].

The first condition required in our construction is that (q, r) can be written as $q=\sigma/\tau$ and $r=\rho/\tau$ with the Hirota dependent variables (σ, τ, ρ) *polynomials* in $x, t_2, \ldots, t_m$. Here, as usual, t_n denotes the time variable associated to the nth equation in the AKNS hierarchy. A family of triples (σ, τ, ρ) that fits in this framework will be presented. This family was found in [22] not in connection with the bispectral property but rather in the study of rational solutions of the integrable Boussinesq system. The second condition in our construction is that for the value $t_2, \ldots, t_m$ under consideration σ and τ, or ρ and τ, have disjoint roots. For the family of examples mentioned above this is a generic property [22, p. 16].

Our result also holds when we restrict to the mKdV hierarchy. This in turn will allow us to connect the present work with the bispectral property for the Schrödinger operator, which was studied in [9].

The plan for this article is the following: In the next section we survey some of the main results and terminology that will be relevant in the rest of the paper. This is mostly intended to provide the nonspecialist in completely integrable systems an understanding of the terminology used and to familiarize the reader with the results in [9]. In Section 3 we prove a proposition that will be employed to establish the main result, which in turn is proven in Section 4. In Section 5 we describe a family of rational solutions of the AKNS hierarchy that fits in the framework of Section 4. Some additional features of the bispectral property for the rational solutions of the mKdV hierarchy are studied in Section 6. In Section 7 we give a general picture of the relations between this article's results and those in [9, 27].

We close this introduction with a few words about notation. We denote by H, E, and F,

$$H = \mathrm{diag}[1, -1]$$

and

$$E = F^{\top} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

As usual, the Wronskian of $f_1, ..., f_n$ will be denoted by $W[f_1, ..., f_n]$. If $f(x, t_2, ..., t_m)$ is a polynomial in x, then $\deg f$ is the degree of f in x. The natural logarithm is denoted by log, and $\partial_x \log f$ is a symbol for $\partial_x f/f$.

2. Preliminaries

One of the first crucial steps in the theory of solitons was the discovery that the Korteweg–de Vries equation

$$v_t = \tfrac{1}{4}(v_{xxx} - 6vv_x)$$

can be written as the compatibility condition of two linear equations, namely, the Schrödinger equation

$$(-\partial_x^2 + v)\varphi = k\varphi \tag{4}$$

and

$$\varphi_t = \tfrac{1}{4}(4\partial_x^3 - 6v\partial_x - 3v_x)\varphi,$$

where k is independent of t.

Such interpretation led to the construction of a whole hierarchy of equations, which arise as the compatibility of (4) and

$$\varphi_t = (c_{2n+1}\partial_x^{2n+1} + \cdots + c_0)\varphi.$$

A number of remarkable facts is associated to this hierarchy of equations. To cite just a few, the initial value problem can be solved by the Inverse Scattering Method [11, 2, 21, 8]. These equations are all Hamiltonian (even bi-Hamiltonian) and the corresponding flows commute. See [18, 20, 2, 25, 24, 21] for further information and for a more comprehensive list of references. These Hamiltonian vector fields restrict in a natural way to certain invariant manifolds, like the N-soliton and the rational manifolds. We call the manifold of rational solutions the set of all rational functions, decaying at infinity, that stay rational by the flow of the KdV equation. It is in fact the union of infinitely many finite dimensional manifolds each one also invariant by the KdV flows.

In [5, 4, 3] the rational solutions of KdV were studied. In [3] it was shown that such solutions can be obtained by finitely many applications of a Darboux transformation [6] starting at $v = 0$. By a Darboux transformation we mean the map

$$v \mapsto \tilde{v} = v - 2\partial_x^2 \log \phi,$$

where ϕ is a nonzero solution of $-\partial_x^2\phi + v\phi = 0$. This transformation is induced by the process of factoring the operator

$$L = -\partial_x^2 + v = -(\partial_x + \phi'/\phi)(\partial_x - \phi'/\phi)$$

and defining a new operator

$$\tilde{L} = -\partial_x^2 + \tilde{v} = -(\partial_x - \phi'/\phi)(\partial_x + \phi'/\phi).$$

If v and $\tilde{v}$ are rational functions of x the transformation will be called a *rational* Darboux transformation. It is a consequence of the results in [4, 3] that the rational solutions of KdV can be written as

$$v = -2\partial_x^2 \log \vartheta_n,$$

where ϑ_n is given by a polynomial in x and in some canonical time variables $p_3, p_5, \ldots, p_{2n+1}$. The polynomial ϑ_n is now called the nth Adler–Moser polynomial.

Let us now describe the interesting connection between the bispectral property and the rational solutions of KdV. Recall that we say that $v(x)$

has the bispectral property if there exists $\varphi(x, k)$, not identically zero, satisfying

$$-\partial_x^2 \varphi + v(x)\varphi = k\varphi, \tag{5}$$

a differential operator $B(k, \partial_k)$, independent of x, and a nonconstant function $\Theta(x)$ such that

$$B(k, \partial_k)\varphi = \Theta(x)\varphi. \tag{6}$$

In [9] it was shown that the bispectral potentials $v(x)$ are necessarily rational functions of x and can be characterized by the following:

(a) If $v(\infty) = \infty$, then $v(x) = ax + b$ for some constants a and b.

(b) Suppose that $v(x)$ is bounded at infinity, then:

— If the dimension of the common space of solutions of (5) and (6) is two for some pair (B, Θ), then, modulo translation and addition of a constant, $v(x)$ is given by $v(x) = c/x^2$ or is obtained by finitely many rational Darboux transformations from $v(x) = -1/4x^2$.

— If the common space of solutions of (5) and (6) is at most one dimensional, then $v(x)$ can be obtained by applying finitely many rational Darboux transformations to $v = 0$.

Moreover, any potential obtained by applying rational Darboux transformations to $v = 0$ or $v = -1/4x^2$ is bispectral.

Another crucial step in the theory of solitons was the interpretation of a number of important equations in Mathematical Physics as the compatibility condition of [1, 26]

$$\partial_x \Psi = (kH + qE + rF)\Psi \stackrel{\text{def}}{=} (kH + Q)\Psi \tag{7}$$

and

$$\partial_{t_n} \Psi = (k^n H + R_1 k^{n-1} + \cdots + R_n)\Psi \stackrel{\text{def}}{=} R^{(n)}\Psi. \tag{8}$$

If we take $n = 1$, we obtain $\partial_{t_1} q = \partial_x q$ and $\partial_{t_1} r = \partial_x r$, which are trivially solved in terms of translations of q and r. This explains our convention of identifying the variable t_1 with x. The first interesting example occurs if we take $n = 2$, $R_1 = qE + rF$, and $R_2 = -(qrH - q_x E + r_x F)/2$. In this case, the compatibility condition of (7) and (8) is

$$\begin{aligned} q_{t_2} &= \tfrac{1}{2}(q_{xx} - 2q^2 r) \\ r_{t_2} &= -\tfrac{1}{2}(r_{xx} - 2r^2 q). \end{aligned} \tag{9}$$

If we specialize to $q = \pm r^*$, and change the time t_2 to it, we obtain the nonlinear Schrödinger equation. Before we continue, an important note for the readers of [20, 10]: the spectral parameter used therein is $\zeta = ik$. The time variables t_j and coefficient R_l used here relate to the time variable t'_j and coefficient Q_j in [20, 10] by

$$t_j = i^{j-1} t'_j \tag{10}$$

and

$$R_l = i^{1-l} Q_l. \tag{11}$$

The following facts are relevant for what follows and can be found in [20, 10]:

(1) Consider the compatibility condition of (7) and (8)

$$\partial_{t_n} Q - \partial_x R^{(n)} + [kH + Q, R^{(n)}] = 0, \tag{12}$$

viewed as an identity in k. For each n, there is a sequence $R_1, ..., R_n$ of matrices whose entries are polynomials in q, r and their derivatives, such that (12) reduces only to the coefficient of k^0. This sequence is uniquely determined once we assign weight $i+1$ to $\partial_x^i q$ and $\partial_x^i r$ and require R_i to be homogeneous of weight i. Moreover, the sequence of matrix polynomials $\{R^{(n)}\}_{n=1}^{\infty}$ so obtained satisfies

$$R^{(n+1)} = kR^{(n)} + R_{n+1}.$$

(2) The nth flow of the AKNS hierarchy is the system obtained by comparing the coefficient of k^0 in (12)

$$\partial_{t_n} Q - \partial_x R_n + [Q, R_n]. \tag{13}$$

(3) The flows generated by these nonlinear evolution equations are Hamiltonian and commute with one another. Hence it makes sense to talk about q and r as functions of $x = t_1, t_2, ..., t_m$ for arbitrarily large m. One example is Eq. (9) given above. Another one is

$$q_{t_3} = \tfrac{1}{4}(q_{xxx} - 6qrq_x)$$
$$r_{t_3} = \tfrac{1}{4}(r_{xxx} - 6qrr_x)$$

which, for $q = r$, yields the mKdV equation.

We conclude this section with a lemma, which is more or less implicit in [20, 10]. This lemma will give a recursion relation to compute the coefficients R_l, and will be used in Sections 5 and 7. If we write $R_l \stackrel{\text{def}}{=} e_l E + f_l F + h_l H$, $e_0 = f_0 = h_1 = 0$, $e_1 = q$, and $f_1 = r$, then Eq. (12) is equivalent to

$$e_{l+1} = qh_l + \tfrac{1}{2}\partial_x e_l \tag{14}$$

$$f_{l+1} = rh_l - \tfrac{1}{2}\partial_x f_l \tag{15}$$

$$\partial_x h_{l+1} = f_{l+1} q - e_{l+1} r \tag{16}$$

with $0 \leqslant l \leqslant n-1$, together with

$$\partial_{t_n} q = \partial_x e_n + 2qh_n \tag{17}$$

$$\partial_{t_n} r = \partial_x f_n - 2rh_n. \tag{18}$$

We consider e_j, h_j, and f_j as polynomials in the variables $\partial_x^l q$ and $\partial_x^l r$, with $l \in \mathbb{Z}_{>0}$.

LEMMA 1. *For* $1 \leqslant j \leqslant n-1$, h_{j+1} *defined by*

$$h_{j+1} = -\tfrac{1}{2} \sum_{l+s=j+1 \ \ l,s \geqslant 1} e_l f_s + h_l h_s \tag{19}$$

satisfies Eq. (16) *with* $l = j$. *Moreover, this choice of* h_{j+1} *is homogeneous of weight* $j+1$ *if we give* $\partial_x^l q$ *and* $\partial_x^l r$ *weight* $l+1$.

Proof. Suppose that Eqs. (14), (15), and (16) hold for $1 \leqslant l \leqslant j-1$, and that (14) and (15) hold for $l = j$. Thus, with prime $= \partial_x$,

$$\begin{aligned}\partial_x h_{j+1} &= -\tfrac{1}{2} \sum_{l+s=j+1 \ \ l,s \geqslant 1} (e_l' f_s + e_l f_s' + h_l' h_s + h_l h_s') \\ &= -\tfrac{1}{2} \sum_{l+s=j+1 \ \ l,s \geqslant 1} 2(e_{l+1} f_s - e_l f_{s+1}) \\ &\quad - 2e_1 h_l f_s + 2f_1 e_l h_s + h_l' h_s + h_l h_s'.\end{aligned}$$

Now we note that

$$\sum_{l+s=j+1 \ \ l,s \geqslant 1} -2e_1 h_l f_s + 2f_1 e_l h_s + h_l' h_s + h_l h_s' = 0$$

because of (16) for $1 \leqslant l \leqslant j-1$. Hence,

$$\begin{aligned}\partial_x h_{j+1} &= \sum_{l+s=j+1 \ \ l,s \geqslant 1} e_{l+1} f_s - e_l f_{s+1} \\ &= e_1 f_{j+1} - e_{j+1} f_1.\end{aligned}$$

To show that the weight of h_{j+1} is $j+1$ we just use induction on j in Eqs. (14), (15), and (16). Q.E.D.

It follows as a consequence of this lemma that the AKNS equations can be written down using the recursive formulae (14), (15) with $l=j$ and (19). Furthermore, if we define e_l, f_l, h_l for all values of $l \in \mathbb{Z}_{>0}$, the nth equation in the AKNS hierarchy can be written as

$$\begin{aligned} q_{t_n} &= 2e_{n+1}(q, ..., \partial_x^n q; r, ..., \partial_x^{n-1} r) \\ r_{t_n} &= -2f_{n+1}(q, ..., \partial_x^{n-1} q; r, ..., \partial_x^n r). \end{aligned} \tag{20}$$

At this point we introduce the τ-function, which plays a key role in the study of completely integrable systems [12, 16, 7, 17, 14, 15]. It is defined as a solution $\tau(x, t_2, ...)$ of

$$h_{j+1} = \tfrac{1}{2} \partial_{t_j} \partial_{t_1} \log \tau, \tag{21}$$

for $j \in \mathbb{Z}_{>0}$. We remark that at least formally this definition makes sense, since $\partial_{t_l} h_{j+1} = \partial_{t_j} h_{l+1}$, as shown in [20]. Together with the introduction of τ, it is natural to define σ and ρ by

$$q = \sigma/\tau \tag{22}$$

and

$$r = \rho/\tau. \tag{23}$$

If we recall that $x = t_1$, Eq. (21) gives for $j=1$,

$$\partial_x^2 \log \tau = -qr. \tag{24}$$

It is easy to see that with this subsitution, (9) becomes

$$\begin{aligned} \sigma_{t_2}\tau - \sigma\tau_{t_2} &= \tfrac{1}{2}(\sigma_{xx}\tau - 2\sigma_x\tau_x + \sigma\tau_{xx}) \\ \rho_{t_2}\tau - \rho\tau_{t_2} &= -\tfrac{1}{2}(\rho_{xx}\tau - 2\rho_x\tau_x + \rho\tau_{xx}). \end{aligned} \tag{25}$$

Using the Hirota operators $D_x, D_{t_2}, D_{t_3}, ..., D_{t_m}$ defined for a polynomial $p(\zeta_1, \zeta_2, ..., \zeta_m)$ by

$$\begin{aligned} &p(D_x, D_{t_1}, ..., D_{t_m}) f \cdot g \\ &\quad \stackrel{\text{def}}{=} p(\partial_{u_1}, \partial_{u_2}, ..., \partial_{u_m}) f(x+u_1, t_2+u_2, ..., t_m+u_m) \\ &\qquad \cdot g(x-u_1, t_2-u_2, ..., t_m-u_m)|_{u_1=u_2=\cdots=u_m=0}, \end{aligned}$$

we have a shorthand for (25)

$$\begin{aligned} D_{t_2}\sigma \cdot \tau - \tfrac{1}{2} D_x^2 \sigma \cdot \tau = 0 \\ D_{t_2}\rho \cdot \tau + \tfrac{1}{2} D_x^2 \rho \cdot \tau = 0. \end{aligned}$$

Equation (24) can be rewritten as

$$D_x^2 \tau \cdot \tau = -2\sigma\rho.$$

DEFINITION 2. If (q, r) satisfies the AKNS hierarchy and (σ, τ, ρ) are defined by Eqs. (21, (22), and (23), then we shall say that σ, τ, and ρ are the Hirota variables associated to (q, r).

3. A SUFFICIENT CONDITION FOR BISPECTRALITY

In this section we prove a basic result that will enable us to establish the bispectrality of certain families of rational potentials satisfying the AKNS hierarchy. This is accomplished by taking into account the form of the eigenfunctions of the corresponding AKNS/ZS operators.

The basic idea is that if (q, r) is in a certain class of rational solutions of the AKNS hierarchy, then we shall prove in Section 4 that a fundamental solution $\Psi(x, k) = \hat{\Psi}(x, k) \exp(kHx)$ of

$$\partial_x \Psi = (kH + qE + rF)\Psi$$

has the form

$$\hat{\Psi}(x, k) = I + \frac{1}{\omega(x)} \begin{bmatrix} p_1^-(x, k) & p_1^+(x, k) \\ p_2^-(x, k) & p_2^+(x, k) \end{bmatrix}, \tag{26}$$

where ω is a polynomial in x and $p_i^\pm$ is a polynomial in x and $1/k$. We write

$$p_i^\pm(x, k) = \sum_{l=1}^{N} p_{i,l}^\pm(x) \frac{1}{k^l}. \tag{27}$$

Moreover, for the eigenfunctions of interest here, we have

$$\deg(p_{i,l}^-) \leqslant \mu^- \stackrel{\text{def}}{=} \max\{\deg(\omega), \deg(p_{2,1}^-)\} \tag{28}$$

and

$$\deg(p_{i,l}^+) \leqslant \mu^+ \stackrel{\text{def}}{=} \max\{\deg(\omega), \deg(p_{1,1}^+)\}. \tag{29}$$

In this situation, we shall show that the bispectrality can be recast in terms of the existence of solutions to certain systems of linear equations. As will be described in Section 4 this situation is typical when we can write

$$\hat{\Psi} = \frac{1}{\tau} \begin{bmatrix} \hat{X}_{-}\tau & -\frac{1}{2k}\hat{X}_{+}\sigma \\ \frac{1}{2k}\hat{X}_{-}\rho & \hat{X}_{+}\tau \end{bmatrix},$$

where

$$\hat{X}_{\pm} = \exp\left(\pm \sum_{j \geq 1} \frac{1}{2jk^j} \frac{\partial}{\partial t_j}\right)$$

and σ, τ, and ρ are polynomials in $t_1 = x, t_2, ..., t_m$, for some m.

Let us write

$$\hat{\Psi} = [\hat{\Psi}^-, \hat{\Psi}^+] = \begin{bmatrix} \hat{\Psi}_1^- & \hat{\Psi}_1^+ \\ \hat{\Psi}_2^- & \hat{\Psi}_2^+ \end{bmatrix}. \tag{30}$$

In the context of Eq. (26), for $k \neq 0$, the question of existence of a differential operator $B^-(k, \partial_k)$ such that $B^- \Psi^- = \Theta(x)\, \Psi^-$ is equivalent to the existence of $\{a_j^-(k), b_j^-(k)\}_{j=0}^m$, independent of x, such that

$$\sum_{j=0}^m a_j^-(\partial_k + x)^j\, \hat{\Psi}_1^- + \sum_{j=0}^m b_j^-(\partial_k + x)^j\, k\hat{\Psi}_2^- = \Theta(x)\, \hat{\Psi}_1^-, \tag{31}$$

and of $\{\tilde{a}_j^-(k), \tilde{b}_j^-(k)\}_{j=0}^m$, also independent of x, such that

$$\sum_{j=0}^m k^{-1}\tilde{a}_j^-(\partial_k + x)^j\, \hat{\Psi}_1^- + \sum_{j=0}^m k^{-1}\tilde{b}_j^-(\partial_k + x)^j\, k\hat{\Psi}_2^- = \Theta(x)\, \hat{\Psi}_2^-. \tag{32}$$

We call the attention of the reader to the factor k in front of $\hat{\Psi}_2^-$ in Eqs. (31) and (32) since it will be useful in the forthcoming computations. The relation between the coefficients of B^- and

$$\{a_j^-(k), b_j^-(k), \tilde{a}_j^-(k), \tilde{b}_j^-(k)\}_{j=0}^m$$

is linear and invertible for $k \neq 0$.

Similarly, the existence of B^+ such that $B^+ \Psi^+ = \Theta(x)\, \Psi^+$ is equivalent to the existence of $\{a_j^+(k), b_j^+(k), \tilde{a}_j^+(k), \tilde{b}_j^+(k)\}_{j=0}^m$, such that

$$\sum_{j=0}^m k^{-1}a_j^+(\partial_k - x)^j\, k\hat{\Psi}_1^+ + \sum_{j=0}^m k^{-1}b_j^+(\partial_k - x)^j\, \hat{\Psi}_2^+ = \Theta(x)\, \hat{\Psi}_1^+ \tag{33}$$

and

$$\sum_{j=0}^{m} \tilde{a}_j^+ (\partial_k - x)^j k \hat{\Psi}_1^+ + \sum_{j=0}^{m} \tilde{b}_j^+ (\partial_k - x)^j \hat{\Psi}_2^+ = \Theta(x) \hat{\Psi}_2^+. \tag{34}$$

In order to ensure that the operator $B^\pm$ is nondegenerate we have to guarantee that

$$\begin{vmatrix} a_m^\pm & b_m^\pm \\ \tilde{a}_m^\pm & \tilde{b}_m^\pm \end{vmatrix} \neq 0. \tag{35}$$

Let us focus on Eqs. (31) and (32). The reasoning for (33) and (34) is completely analogous. For simplicity we drop the minus superscript.

Equations (31) and (32), in light of (26) and (27), read as

$$\sum_{j=0}^{m} a_j(\partial_k + x)^j \left(\omega + \sum_{l=1}^{N} p_{1,l} k^{-l}\right) + \sum_{j=0}^{m} b_j(\partial_k + x)^j \sum_{l=1}^{N} p_{2,l} k^{-l+1}$$
$$= \Theta(x)\left(\omega + \sum_{l=1}^{N} p_{1,l} k^{-l}\right) \tag{36}$$

and

$$\sum_{j=0}^{m} \tilde{a}_j(\partial_k + x)^j \left(\omega + \sum_{l=1}^{N} p_{1,l} k^{-l}\right) + \sum_{j=0}^{m} \tilde{b}_j(\partial_k + x)^j \sum_{l=1}^{N} p_{2,l} k^{-l+1}$$
$$= \Theta(x)\left(\sum_{l=1}^{N} p_{2,l} k^{-l+1}\right). \tag{37}$$

Let $s \stackrel{\text{def}}{=} \deg(\Theta)$. Equations (36), (37), and the nondegeneracy of B imply that we should take $s = m$, where m is the order of B.

The central point of our approach is the following: Expand (36) and (37) in powers of x and consider for each coefficient the resulting equation. The problem of finding $\{a_j, b_j\}$ such that (36) holds reduces to a problem of solving $m + \mu + 1$ equations in $2m + 2$ unknowns. It is reasonable to expect that solutions of this system exist if $m + 1 \geqslant \mu$. However, this is not evident since the polynomials $p_{i,l}$ may be quite special. Also, we have to guarantee that (35) holds. Proposition 4 will ensure the existence of the coefficients of B, but first we need a very simple algebraic lemma:

LEMMA 3. *Let* $\omega(x) = \omega_\mu x^\mu + \cdots$ *and* $\rho(x) = \rho_\mu x^\mu + \cdots$ *be two polynomials with* $\omega_\mu \neq 0$ *and* $\rho(x) \neq 0$. *If* p_1 *and* p_2 *are in the vector space of polynomials of degree less than or equal to* $\mu - 1$, *then the equation*

$$\omega p_1 + \rho p_2 = 0 \tag{38}$$

has only the trivial solution $p_1 = p_2 = 0$ *iff* $\gcd(\omega, \rho) = 1$.

Proof. Suppose that $\gcd(\omega, \rho) = 1$ and that there exists $(p_1, p_2) \neq (0, 0)$ such that Eq. (38) holds, with $\deg(p_i) \leqslant \mu - 1$. Every root α of ω is also a root of ρp_2. Since $\gcd(\omega, \rho) = 1$, α is a root of p_2 with at least the same multiplicity as in ω. Since $\omega_\mu \neq 0$, ω has μ roots (counting multiplicities) and hence $p_2 = 0$. This implies that $p_1 = 0$, a contradiction. If $\gcd(\omega, \rho) \neq 1$ there exists a common root α of ω and ρ. Take $p_1 = \rho/(x - \alpha)$, $p_2 = \omega/(x - \alpha)$ and note that (p_1, p_2) is a nontrivial solution of (38) with $\deg(p_i) \leqslant \mu - 1$. Q.E.D.

PROPOSITION 4. *Suppose that Eqs.* (28) *and* (29) *are satisfied. If* $\gcd(\omega, p_{2,1}^-) = 1$, *then for any polynomial* $\Theta(x)$ *of degree greater than or equal to* $\mu^- = \max\{\deg(\omega), \deg(p_{2,1}^-)\}$ *there exists a nondegenerate operator* $B^-(k, \partial_k)$, *such that*

$$B^-(k, \partial_k)\, \Psi^- = \Theta(x)\, \Psi^-,$$

with Ψ^- *as in Eqs.* (30) *and* (26).

If $\gcd(\omega, p_{1,1}^+) = 1$, *then for any polynomial* $\Theta(x)$ *of degree greater than or equal to* $\mu^+ = \max\{\deg(\omega), \deg(p_{1,2}^+)\}$ *there exists a nondegenerate operator* $B^+(k, \partial_k)$, *such that*

$$B^+(k, \partial_k)\, \Psi^+ = \Theta(x)\, \Psi^+.$$

Proof. For brevity we consider only the minus case, the plus one is entirely analogous. This allows us to drop the minus superscript. The idea of the proof is to choose $a_m, a_{m-1}, ..., a_\mu, b_m, b_{m-1}, ..., b_\mu, \tilde{a}_m, \tilde{a}_{m-1}, ..., \tilde{a}_\mu$, and $\tilde{b}_m, \tilde{b}_{m-1}, ..., \tilde{b}_\mu$ in a suitable way, taking advantage of the underdeterminacy of the systems associated to Eqs. (36) and (37). Then, we use the assumption $\gcd(\omega, p_{2,1}^\pm) = 1$ to ensure that the remaining coefficients $a_{\mu-1}, ..., a_0, b_{\mu-1}, ..., b_0, \tilde{a}_{\mu-1}, ..., \tilde{a}_0, \tilde{b}_{\mu-1}, ..., \tilde{b}_0$ can be found in a unique way.

Let $\rho(x) = p_{2,1}^-(x)$ and $\Theta(x) = \Theta_0 x^m + \cdots \Theta_m$, with $\Theta_0 \neq 0$ and $m \geqslant \mu$. Choose $a_m = \tilde{b}_m = \Theta_0$ and $\tilde{a}_m = b_m = 0$, which will ensure the nondegeneracy of B. Equation (36) reduces to

$$\sum_{j=0}^{m-1} a_j (\partial_k + x)^j \left(\omega + \mathcal{O}_1\left(x, \frac{1}{k}\right)\right) + \sum_{j=0}^{m-1} b_j (\partial_k + x)^j \left(\rho + \mathcal{O}_2\left(x, \frac{1}{k}\right)\right) = P_{m-1}\left(x, \frac{1}{k}\right),$$

where $P_{m-1}(x, 1/k)$ is a polynomial in x and $1/k$ of degree less than $\mu + m - 1$ in x. For $l = 1, 2$, the symbol $\mathcal{O}_l$ refers to a polynomial in x and $1/k$ of degree less than μ in x, and satisfying $\lim_{k \to \infty} \mathcal{O}_l(x, 1/k)) = 0$.

Similarly,

$$\sum_{j=0}^{m-1} \tilde{a}_j(\partial_k + x)^j \left(\omega + \mathcal{O}_1\left(x, \frac{1}{k}\right)\right) + \sum_{j=0}^{m-1} \tilde{b}_j(\partial_k + x)^j \left(\rho + \mathcal{O}_2\left(x, \frac{1}{k}\right)\right) = \tilde{P}_{m-1}\left(x, \frac{1}{k}\right), \tag{39}$$

with $\deg(\tilde{P}_{m-1}) \leqslant \mu + m - 1$. This shows for $i = 0$ the following claim, which we shall prove for any $i \leqslant m - \mu$:

We can choose a_{m-i}, b_{m-i}, $\tilde{a}_{m-i}$ and $\tilde{b}_{m-i}$, so that (36) reduces to

$$\sum_{j=0}^{m-i} a_j^+(\partial_k + x)^j \left(\omega + \mathcal{O}_1\left(x, \frac{1}{k}\right)\right) + \sum_{j=0}^{m-i} b_j^+(\partial_k + x)^j \left(\rho + \mathcal{O}_2\left(x, \frac{1}{k}\right)\right) = P_{m-i}\left(x, \frac{1}{k}\right),$$

where $P_{m-i}(x, 1/k)$ is a polynomial in x and $1/k$ of degree less than $\mu + m - i$ and a similar statement holds for Eq. (37). This is done using induction as follows: by assumption $(\omega, \rho) = (\omega_\mu, \rho_\mu)\, x^\mu + \cdots$, with $(\omega_\mu, \rho_\mu) \neq 0$. Suppose that $\omega_\mu \neq 0$. A completely analogous argument holds if $\rho_\mu \neq 0$. Assume that the claim holds up to $i \leqslant m - \mu$, let us prove that it holds for $i + 1$. For this just note that

$$(\partial_k + x)^j \left(\omega + \mathcal{O}_1\left(x, \frac{1}{k}\right)\right) = (\omega_\mu + h(1/k))\, x^{\mu + j} + \eta(x, 1/k),$$

where h is a polynomial with $h(0) = 0$ and $\deg(\eta(x, 1/k)) < \mu + i - 1$. (Here we are using the assumption that $\deg(p_{i,j}) \leqslant \mu$.) So if we set $b_{m-i} = 0$ and take $a_{m-i}(k)$ equal to the coefficient of degree $m - i + \mu$ in x of P_{m-i} divided by $(\omega_\mu + h(1/k))$, we obtain that

$$P_{m-i} - a_{m-i}(\partial_k + x)^{m-i}(\omega + \mathcal{O}_1) - b_{m-i}(\partial_k + x)^{m-j}(\rho + \mathcal{O}_2)$$

has degree $\mu + m - i - 1$. We apply the same recipe to the equations corresponding to (39). We stop at $i = m - \mu$, since now we cannot take coefficients arbitrarily, and we prove that the systems associated to

$$\sum_{j=0}^{\mu-1} a_j(\partial_k + x)^j \left(\omega + \mathcal{O}_1\left(x, \frac{1}{k}\right)\right) + \sum_{j=0}^{\mu-1} b_j(\partial_k + x)^j \left(\rho + \mathcal{O}_2\left(x, \frac{1}{k}\right)\right) = P_{\mu-1}\left(x, \frac{1}{k}\right) \tag{40}$$

and

$$\sum_{j=0}^{\mu-1} \tilde{a}_j(\partial_k + x)^j \left(\omega + \mathcal{O}_1\left(x, \frac{1}{k}\right)\right) + \sum_{j=0}^{\mu-1} \tilde{b}_j(\partial_k + x)^j \left(\rho + \mathcal{O}_2\left(x, \frac{1}{k}\right)\right) = \tilde{P}_{\mu-1}\left(x, \frac{1}{k}\right). \tag{41}$$

Can be uniquely solved. Our construction ensures that the right hand side of (40) and (41) have degree in x strictly less than 2μ. To prove the existence and uniqueness of $a_0, ..., a_{\mu-1}, b_0, ..., b_{\mu-1}$, we note that all we need to do is to ensure that (40) and (41) have only the trivial solution if we take the right hand side $P_{\mu-1} = \tilde{P}_{\mu-1} = 0$. Since the determinant of the corresponding system is a meromorphic function of k, this result follows if we show that such property holds for $k = \infty$. But when $k = \infty$ Eq. (40) with zero r.h.s. reduces to

$$\left(\sum_{j=0}^{\mu-1} a_j x^j\right) \omega + \left(\sum_{j=0}^{\mu-1} b_j x^j\right) \rho = 0, \tag{42}$$

and the same is true for (41). Lemma 3 implies that the only solution $a_0, ..., a_{\mu-1}, b_0, ..., b_{\mu-1}$ of (42) is the trivial one. Q.E.D.

4. Rational Solutions of AKNS and Bispectrality

The first objective of this section is to prove the main result of this article, which establishes the bispectrality of the operators of the form $\mathbf{L} = H(\partial_x - qE - rF)$ with q and r certain potentials in the manifold of rational solutions of the AKNS hierarchy. The second objective is to prove an amplification this result. This allows us to restrict our attention to finitely many flows of the AKNS hierarchy, as long as (q, r) is a stationary solution of a certain flow of sufficiently high order.

Let us start by remarking that the key object in obtaining the bispectral eigenfunctions we are interested on will be formal operators $X_{\pm}$, acting on a function f of infinitely many variables $t_1, ..., t_j, ...$ by the formula

$$X_{\pm} f(t_1, t_2, ...) = \exp\left(\mp \sum_{j=1}^{\infty} k^j t_j\right) \hat{X}_{\pm} f(t_1, t_2, ...) \tag{43}$$

$$= \exp\left(\mp \sum_{j=1}^{\infty} k^j t_j\right) \exp\left(\pm \sum_{j\geqslant 1} \frac{1}{2jk^j} \frac{\partial}{\partial t_j}\right) f(t_1, t_2, ...) \tag{44}$$

$$= \exp\left(\mp \sum_{j=1}^{\infty} k^j t_j\right) f\left(t_1 \pm \frac{1}{2k}, t_2 \pm \frac{1}{4k^2}, ...\right). \tag{45}$$

Of course, these objects make perfect sense if the function f depends only on finitely many variables and we take $t_j=0$ for all but finitely many values of j. In what follows we shall keep this convention in mind.

THEOREM 5. *Let (q, r) be a rational solution of the AKNS hierarchy such that the associated Hirota variables σ, τ, and ρ are polynomials in $t_1=x$, $t_2, ..., t_m$. Suppose $t_2, ..., t_m$ is such that σ and τ have disjoint roots. Then, given any polynomial $\Theta(x)$, satisfying* $\deg(\Theta)\geqslant \max\{\deg(\sigma), \deg(\tau)\}$, *there exists a nondegenerate operator $B^-(k, \partial_k)$, independent of x, such that*

$$B^-(k, \partial_k)\, \Psi^- = \Theta(x)\, \Psi^-,$$

where Ψ^- is the eigenfunction of

$$\mathbf{L} = H(\partial_x - qE - rE)$$

given by

$$\Psi^- = \frac{1}{\tau}\begin{bmatrix} X_-\tau \\ \frac{1}{2k} X_-\rho \end{bmatrix}. \tag{46}$$

Suppose $t_2, ..., t_m$ is such that ρ and τ have disjoint roots. Then, given any polynomial $\Theta(x)$, satisfying $\deg(\Theta)\geqslant \max\{\deg(\rho), \deg(\tau)\}$, *there exists $B^+(k, \partial_k)$ such that*

$$B^+(k, \partial_k)\, \Psi^+ = \Theta(x)\, \Psi^+,$$

where Ψ^+ is the eigenfunction of $\mathbf{L}$ *given by*

$$\Psi^+ = \frac{1}{\tau}\begin{bmatrix} -\frac{1}{2k} X_+\sigma \\ X_+\tau \end{bmatrix}. \tag{47}$$

Proof. We first show that $\mathbf{L}\Psi^\pm = k\Psi^\pm$. Since σ, τ, and ρ are polynomials in the variables $t_1, ..., t_m$, we have that

$$\hat{X}_\pm \tau = \exp\left(\pm \sum_{j\geqslant 1} \frac{1}{2jk^j} \frac{\partial}{\partial t_j}\right) \tau,$$

$\hat{X}_\pm\sigma$, and $\hat{X}_\pm\rho$ are also polynomials in the variables $t_1, t_2, ..., t_m, 1/k$. The assumptions on (σ, τ, ρ) imply that there exists a fundamental system of common solutions of Eq. (8) for an arbitrary number of values of n. According to the results of [20, p. 174], there exists a formal fundamental solution $\Psi = [\Psi^-, \Psi^+]$ with $\Psi^\pm$ as in Eqs. (46) and (47). Hence, as a formal power series in $1/k$, the expression $\partial_x \Psi - (kH + qE + rF)\Psi$ is identically zero. But since

$$\tau^2(\partial_x \Psi - (kH + qE + rF)\Psi) \exp\left(-\sum_{j \geqslant 1} t_j k^j\right)$$

is a polynomial in $t_1, ..., t_m$, $1/k$, it follows that Ψ is a true solution of $\partial_x \Psi = (kH + qE + rF)\Psi$.

To establish the existence of $B^\pm$ is equivalent to showing that there exists $A^\pm(k, \partial_k)$ such that

$$A^\pm(k, \partial_k)\left(\exp\left(\pm\sum_{j=2}^{\infty} k^j t_j\right)\Psi^\pm\right) = \Theta(x)\left(\exp\left(\pm\sum_{j=2}^{\infty} k^j t_j\right)\Psi^\pm\right).$$

Now we note that the function

$$\hat{\Psi} = \left[\Psi^- \exp\left(-\sum_{j=1}^{\infty} k^j t_j\right), \Psi^+ \exp\left(\sum_{j=1}^{\infty} k^j t_j\right)\right]$$

has the form (26) with $\omega = \tau$. We expand $\hat{\Psi} = [\hat{\Psi}^-, \hat{\Psi}^+]$, in Laurent series at $k = 0$, and we define the coefficients $p_{i,l}^\pm$ by Eq. (26). Using the same notation of Section 3 we claim that

$$\deg(p_{i,l}^-) \leqslant \max\{\deg(\tau), \deg(\rho)\} \tag{48}$$

and

$$\deg(p_{i,l}^+) \leqslant \max\{\deg(\tau), \deg(\sigma)\}. \tag{49}$$

To see this just note that if $g(t_1, t_2, ..., t_m)$ is a polynomial, then

$$g\left(t_1 + \lambda, t_2 + \frac{1}{2}\lambda^2, ..., t_m + \frac{1}{m}\lambda^m\right)$$
$$= g(t_1, t_2, ..., t_m) + \sum_{j \geqslant 1} \sum_{\alpha = (\alpha_1, ..., \alpha_m)} \lambda^j \gamma_{j,\alpha} \partial_{t_1}^{\alpha_1} \cdots \partial_{t_m}^{\alpha_m} g,$$

where $\gamma_{j,\alpha}$ is a universal constant and the inner sum is taken over all $\alpha_1, ..., \alpha_m \geqslant 0$, satisfying $1\alpha_1 + 2\alpha_2 + \cdots + m\alpha_m = j$. Equations (48) and (49) follow from the fact that $\deg(\partial_{t_1}^{\alpha_1} \cdots \partial_{t_m}^{\alpha_m} g) \leqslant \deg(g)$. In the present case $p_{2,1}^- = \rho/2$ and $p_{1,1}^+ = -\sigma/2$. The result now follows from Proposition 4.

Q.E.D.

Remarks. (1) A similar result should hold for rational solutions of systems obtained by isomonodromic deformations, as described in [17], Example 6.2.

(2) The condition $\gcd(\sigma, \tau) = 1$ might be unnecessary, given the experience we have with the rational solutions of the mKdV hierarchy. In this case,

$$q = r = \partial_x \log(\vartheta_{n+1}/\vartheta_n) = (\vartheta'_{n+1}\vartheta_n - \vartheta'_n\vartheta_{n+1})/(\vartheta_n\vartheta_{n+1}),$$

where ϑ_j is the jth Adler–Moser polynomial [3]. If ϑ_n (or ϑ_{n+1}) has a double root then $\gcd((\vartheta_n\vartheta_{n+1}), (\vartheta'_{n+1}\vartheta_n - \vartheta'_n\vartheta_{n+1})) \neq 1$. On the other hand, in this case it is shown in [27] that there exist bispectral eigenfunctions for any $\Theta(x)$ that is divisible by $\vartheta_n\vartheta_{n+1}$. This leads us to conjecture that it might be possible to substitute the condition $\gcd(\sigma, \tau) = 1$ or $\gcd(\rho, \tau) = 1$ by restrictions on the function $\Theta(x)$.

(3) Another example is the case $\rho = x^3/6$, $\tau = x^4/12$, and $\sigma = x^3/6$. In this case the bispectrality is evident although the gcd condition is not satisfied. We show in Section 7 that

$$\begin{aligned} \rho &= (2x^3 + 6t_2x + 3t_3)/12 \\ \tau &= -(x^4 - 3t_3x + 3t_2^2)/12 \\ \sigma &= (2x^3 - 6t_2x + 3t_3)/12 \end{aligned}$$

are Hirota dependent variables associated to certain solutions of the AKNS hierarchy. (They are obtained by taking $n = 3$ and $j = 1$ in Eq. (53) of Section 7.) On the other hand, a straightforward computation shows that the gcd condition is satisfied generically in the time variables t_2 and t_3. Indeed the resultant of τ and ρ is, modulo a multiplicative constant,

$$(9t_3^2 + 16t_2^3)^2,$$

and the resultant of τ and σ is, modulo a multiplicative constant,

$$(9t_3^2 - 16t_2^3)^2.$$

(4) If we consider $\Theta(x)$ such that $\deg(\Theta) = \mu - 1$, the construction of Proposition 4 still gives the existence of $B^\pm$ satisfying $B^\pm\Psi^\pm = \Theta(x)\Psi^\pm$, as long as the gcd condition is satisfied. However, in this case it is not clear that $B^\pm$ is nondegenerate.

(5) Another interesting question is whether or not there exists $\Theta(x)$ and a nondegenerate $B(k, \partial_k)$ such that for some *eigenmatrix* $\Psi(x, k)$ of $\mathbf{L}$ we have $B(k, \partial_k)\Psi = \Theta(x)\Psi$. This would correspond to the rank two case in the terminology of [9, 27]. In [28] we have shown how to construct

nontrivial bispectral operators of the form $\mathbf{L} = J^{-1}(\partial_x - Q(x))$ for certain matrices J and $Q(x)$. In this case we exhibit an eigenmatrix of $\mathbf{L}$ satisfying $B\Psi = \Theta\Psi$. However, the potentials $Q(x)$ obtained are *not* in the manifold of rational solutions of the associated hierarchy. The picture here is analogous to the one in the Schrödinger case [9], where the rank two bispectral potentials were not in the manifold of rational solutions of the KdV hierarchy.

We shall now relax the hypothesis of Theorem 5 a little bit. It assumes that (q, r) satisfies all AKNS equations and that the associated Hirota variables are polynomials. This in turn implies that for some m:

(i) The pair $(q(x, t_2, ..., t_m), r(x, t_2, ..., t_m))$ satisfies Eqs. (20) for $n = 1, ..., m$.

(ii) The pair $(q(x, t_2, ..., t_m), r(x, t_2, ..., t_m))$ is a stationary solution of (20) for $n = m+1$, i.e., the r.h.s. of (20) vanishes for $n = m+1$.

(iii) There exists a triple of polynomials (σ, τ, ρ) in the variables $x, t_2, ..., t_m$ satisfying

$$\tfrac{1}{2}\,\partial_{t_n}\partial_{t_1} \log \tau = h_{j+1}, \tag{50}$$

for $j = 1, ..., m$, $\sigma = q\tau$, and $\rho = r\tau$.

We shall show that a converse of this implication holds, namely, if properties (i), (ii), and (iii) above are valid, then (q, r) is a solution of all equations in the AKNS hierarchy and that (σ, τ, ρ) are the Hirota variables associated to (q, r). To do this we start with:

LEMMA 6. *If (q, r) are rational functions of x such that $\lim_{x\to\infty} qr = 0$, and e_l, f_l, and h_l are defined by the recursion relations* (14), (15), *and* (19), *with $h_0 = 1$, $e_0 = f_0 = h_1 = 0$, $e_1 = q$, and $f_1 = r$, then*

$$\lim_{x\to\infty} h_j = 0, \qquad \forall j \geqslant 1.$$

Proof. We show by induction that

$$\lim_{x\to\infty} e_l f_s = 0 \qquad \text{if} \quad l+s = j > 1 \text{ and } l, s \geqslant 1$$

$$\lim_{x\to\infty} h_l = 0 \qquad \text{if} \quad 1 \leqslant l \leqslant j.$$

Take integers $l, s \geqslant 1$ such that $l+s = j+1$. If $l > 1$, we claim that $e_l f_s = (e_1 h_{l-1} + \frac{1}{2}\partial_x e_{l-1}) f_s$ goes to zero as $x \to \infty$. Indeed, since we are assuming that q and r are rational in x, $\lim_{x\to\infty} e_{l-1} f_s = 0$ implies that

$f_s\,\partial_x e_{l-1} \to 0$ as $x \to \infty$. If $l=1$ and $s>1$ we apply the same reasoning but with f_s instead of e_l. Hence,

$$\lim_{x\to\infty} h_{j+1} = \lim_{x\to\infty} -\tfrac{1}{2} \sum_{l+s=j+1\ \ l,s\geqslant 1} e_l f_s + h_l h_s = 0.$$

The beginning of the induction and the case $l=s=1$ is obvious, since $h_1 = 0$ and $\lim_{x\to\infty} e_1 f_1 = 0$ by hypothesis. Q.E.D.

This lemma is used to show:

PROPOSITION 7. *Suppose that* $q(x, t_2, ..., t_m)$ *and* $r(x, t_2, ..., t_m)$ *are rational functions of* x *and* $\lim_{x\to\infty} qr = 0$. *If* (q, r) *is a solution of the first* m *flows in the AKNS hierarchy, and if* (q, r) *is a stationary solution of the* $(m+1)$*st one, then* (q, r) *is a stationary solution of all the flows of order greater than* $m+1$. *Furthermore,* $h_l = 0$ *for* $l \geqslant m+2$.

Proof. The assumption that (q, r) is a stationary solution of the $(m+1)$st flow of the AKNS hierarchy means that $e_{m+2} = f_{m+2} = 0$. We claim that this implies that $h_{m+2} = 0$. Indeed, from Eq. (16) with $l = m+1$ we have that $\partial_x h_{m+2} = f_{m+2} e_1 - e_{m+2} f_1 = 0$. Thus h_{m+2} is a constant. From the previous lemma $\lim_{x\to\infty} h_{m+2} = 0$, and so $h_{m+2} = 0$. Hence, Eqs. (14) and (15) with $l = m+2$, imply that $e_{m+3} = f_{m+3} = 0$. Therefore, (q, r) is a stationary solution of the $(m+2)$nd flow. A simple induction proves the result for all the higher order flows. Q.E.D.

We can now state the following amplification of Theorem 5:

THEOREM 8. *Suppose that* (q, r) *satisfies the conditions* (i), (ii), *and* (iii), *and let* (σ, τ, ρ) *be the corresponding Hirota variables. Then, for any value of* $t_2, ..., t_m$, *such that the roots of* σ *and* τ *are disjoint* (*or the roots of* ρ *and* τ) *the operator* $\mathbf{L} = H(\partial_x - qE - rF)$ *is bispectral. Furthermore, bispectral eigenfunctions are given by Eq.* (46) (*respectively by Eq.* (47)).

Proof. All we need to note is that for $l=1$, Eq. (19), implies that $\lim_{x\to\infty} qr = 0$, because τ is assumed to be a polynomial in x. Hence, from Proposition 7, $h_l = e_l = f_l = 0$, for $l \geqslant m+2$, and so Eq. (50) holds for $j \geqslant m+1$. Now we can use Theorem 5 to complete the proof. Q.E.D.

5. SCHUR POLYNOMIALS AND RATIONAL SOLUTIONS OF THE AKNS HIERARCHY

It is clear that the result of Section 4 would not be of much interest if the pairs (q, r) for which it holds were only the trivial ones. Fortunately, this is not the case as we shall see in this section. In fact, a very nice class of rational solutions of the AKNS hierarchy was found by Robert Sachs in [22]. See also earlier work on rational solutions of the nonlinear Schrödinger that was done in [19]. The description given in [22] is mostly in terms of the integrable Boussinesq hierarchy, which is formally equivalent to the AKNS hierarchy [13]. Using our time variables the rational solutions of the AKNS hierarchy found in [22] can be described as follows:

Let $q_j(y_1, ..., y_n)$ be the elementary Schur polynomial defined by

$$\sum_{j \geqslant 0} \lambda^j q_j = \exp\left(\sum_{j \geqslant 1} \lambda^j y_j\right). \tag{51}$$

For convenience we set $q_l = 0$ if $l < 0$. Define

$$y_1 = x = t_1, \quad y_2 = -\frac{1}{2} t_2, \quad y_3 = \frac{1}{4} t_3, \quad ..., \quad y_j = \frac{1}{(-2)^{j-1}} t_j, \tag{52}$$

and take

$$\begin{aligned} \sigma_j &= W[q_n, ..., q_{n-j+1}] \\ \tau_j &= W[q_n, ..., q_{n-j}] \\ \rho_j &= -W[q_n, ..., q_{n-j-1}], \end{aligned} \tag{53}$$

where q_l, for $l \in \mathbb{Z}$, is considered as a function of $x, t_2, t_3, ..., t_n$. The derivatives in the Wronskians are with respect to x. The dependence of $(\sigma_j, \tau_j, \rho_j)$ on n is omitted for notational simplicity. We also extend our definition for $j = 0$ by setting $\sigma_0 = 1$ in this case, and for $j = -1$ by taking $\sigma_{-1} = 0$ and $\tau_{-1} = 1$. Using the Jacobi identity, which states that

$$W[W[f_1, ..., f_n, g], W[f_1, ..., f_n, h]] = W[f_1, ..., f_n]\, W[f_1, ..., f_n, g, h],$$

it follows that, for $-1 \leqslant j \leqslant n$,

$$D_x^2 \tau_j \cdot \tau_j = -2\sigma_j \cdot \rho_j.$$

This was remarked already in [22].

The purposes of this section are the following:

(1) To confirm that if we write $(\sigma_j, \tau_j, \rho_j)$ as in Eqs. (51), (52), and (53), then $(q, r) = (\sigma_j/\tau_j, \rho_j/\tau_j)$ is a solution of the AKNS hierarchy. Furthermore, $(\sigma_j, \tau_j, \rho_j)$ are the Hirota dependent variables associated to (q, r).

(2) To prove directly that a fundamental system of solutions of

$$\partial_x \Psi = \left(\frac{\sigma_j}{\tau_j} E + \frac{\rho_j}{\tau_j} F + kH\right) \Psi$$

can be written as

$$\Psi = \frac{1}{\tau_j}\begin{bmatrix} X_-\tau_j & -\frac{1}{2k}X_+\sigma_j \\ \frac{1}{2k}X_-\rho_j & X_+\tau_j \end{bmatrix}, \tag{54}$$

where $X_\pm$ is given by Eq. (43).

We remark that the first item above was accomplished in [22] by different methods. We start with the following:

LEMMA 9. *If q_n is defined by* (51) *and* (52), *then*

$$\hat{X}_+ q_n = q_n + \frac{1}{2k} q_{n-1}$$

and

$$\hat{X}_- q_n = \sum_{l=0}^{n} \frac{1}{(-2k)^{n-l}} q_{n-l}.$$

Proof. Recall that for $|\xi| < 1$,

$$\sum_{j\geq 1} \frac{(-1)^j}{j} \xi^j = -\log(1+\xi).$$

Now, we compute for $\xi = \lambda/2k$

$$\begin{aligned} &\exp\left(\pm \sum_{j\geq 1} \frac{1}{2jk^j} \partial_{t_j}\right) \exp\left(\sum_{j\geq 1} \lambda^j \frac{1}{(-2)^{j-1}} t_j\right) \\ &= \exp\left(\sum_{j\geq 1} \lambda^j \frac{t_j}{(-2)^{j-1}}\right) \exp\left(\mp \sum_{j\geq 1} \frac{(-1)^j}{j} \xi^j\right) \\ &= \left(1 + \frac{\lambda}{2k}\right)^{\pm 1} \exp\left(\sum_{j\geq 1} \lambda^j \frac{t_j}{(-2)^{j-1}}\right). \end{aligned}$$

We conclude the proof by expanding the l.h.s. and the last r.h.s. of this equation in a power series in the variable λ and comparing the coefficients. Q.E.D.

In order to accomplish the first goal we start by showing the result for $\sigma = 0$, $\tau = 1$, and $\rho = -q_n$. This corresponds to taking $j = -1$ if we use the convention introduced above. But in this case we can easily show that

$e_l = h_l = 0$ for $l \geqslant 1$, and $f_{l+1} = (-2)^{-l} \partial_x^l r$. So, it suffices to show that $\partial_{t_l} r = (-2)^{-l+1} \partial_x^l r$. This is indeed true because

$$\partial_{t_l} q_n = (-2)^{-l+1} q_{n-l} \tag{55}$$

as can be seen by differentiating Eq. (51) with respect to t_l.

One can also show in this case that if we take $h = \sum_{l \geqslant 0} h_l k^{-l}$, $e = \sum_{j \geqslant 1} k^{-j} e_j$, and $f = \sum_{j \geqslant 1} k^{-j} f_j$, then

$$\begin{aligned} \begin{bmatrix} h & e \\ f & -h \end{bmatrix} \begin{bmatrix} X_- \tau & -\dfrac{1}{2k} X_+ \sigma \\ \dfrac{1}{2k} X_- \rho & X_+ \tau \end{bmatrix} \\ = \begin{bmatrix} X_- \tau & -\dfrac{1}{2k} X_+ \sigma \\ \dfrac{1}{2k} X_- \rho & X_+ \tau \end{bmatrix} H. \end{aligned} \tag{56}$$

This equation appears in [20, p. 179] modulo a small typo and the change of variables indicated in Eqs. (10) and (11). Again the verification is straightforward in the case $\sigma = 0$, $\tau = 1$, and $\rho = -q_n$. Indeed, $\hat{X}_{\mp} \tau = 1$, $\hat{X}_+ \sigma = 0$ and using Lemma 9 followed by Eq. (55)

$$\begin{aligned} f &= -\frac{1}{k} q_n + \sum_{l=1}^{\infty} \frac{1}{2} \frac{1}{k^{l+1}} \partial_{t_l} q_n \\ &= -\frac{1}{k} \hat{X}_- q_n. \end{aligned}$$

To obtain the same results mentioned above for triples $(\sigma_j, \tau_j, \rho_j)$, with $0 \leqslant j \leqslant n$, we remark that such triples are obtained by applying $j+1$ Schlesinger transformations, as defined below, to $(0, 1, -q_n)$ followed by the harmless multiplication of τ by -1. This fact about Schlesinger transformations was also pointed out in [22], and can be shown by induction on j with the help of Jacobi's identity. The Schlesinger transformation here is

$$S_- : (\sigma, \tau, \rho) \mapsto (\sigma_-, \tau_-, \rho_-) \stackrel{\text{def}}{=} \left(\tau, \rho, -\frac{1}{2\tau} D_x^2 \rho \cdot \rho \right).$$

It is induced by the gauge transformation

$$\tilde{\Psi} = \begin{bmatrix} 0 & \tau/\rho \\ \rho/\tau & -2k - (\rho_x \tau - \rho \tau_x)/(\rho\tau) \end{bmatrix} \Psi.$$

The first claim now follows from the fact that (56) and the equations in the AKNS hierarchy are invariant by these Schlesinger transformations (see [20, Chap. 5] or [17, 14, 15]).

Let us prove the second item. Here again the proof is simplified if we first show the result for $\sigma=0$, $\tau=1$, and $\rho=-q_n$ or for $\sigma=1$, $\tau=q_n$, and $\rho=-W[q_n, q_{n-1}]$. The eigenfunction $\tilde{\Psi}$ has the form (54) with (σ, τ, ρ) substituted by $(\sigma_-, \tau_-, \rho_-)$ and multiplied on the right by diag$[(1/2k), -2k]$. Using Eq. (54) and writing $(\sigma, \tau, \rho)=(\sigma_j, \tau_j, \rho_j)$ the proof reduces to showing that the following equations are satisfied:

$$-\tau_x \hat{X}_- \tau + \tau \hat{X}_- \tau_x - \frac{1}{2k} \sigma \hat{X}_- \rho = 0, \tag{57}$$

$$\frac{1}{2k}(\tau_x \hat{X}_+ \sigma - \tau \hat{X}_+ \sigma_x) - \sigma \hat{X}_+ \tau + \tau \hat{X}_+ \sigma = 0, \tag{58}$$

$$\frac{1}{2k}(-\tau_x \hat{X}_- \rho + \tau \hat{X}_- \rho_x) - \rho \hat{X}_- \tau + \tau \hat{X}_- \rho = 0, \tag{59}$$

and

$$-\tau_x \hat{X}_+ \tau + \tau \hat{X}_+ \tau_x + \frac{1}{2k} \rho \hat{X}_+ \sigma = 0. \tag{60}$$

The proof of Eqs. (57), (58), (59), and (60) now becomes a straightforward computation for $j=-1$. Let us do (57), in the case $\sigma=1$, $\tau=q_n$, and $\rho=-W[q_n, q_{n-1}]$,

$$\begin{aligned}
&-\tau_x \hat{X}_- \tau + \tau \hat{X}_- \tau_x - \frac{1}{2k} \sigma \hat{X}_- \rho \\
&\quad = -q_{n-1} \sum_{l=0}^{n} (-2k)^{l-n} q_l + q_n \sum_{l=0}^{n-1} (-2k)^{l+1-n} q_{l-1} \\
&\qquad - \frac{1}{2k} W\left[\sum_{l=0}^{n-1} (-2k)^{l+1-n} q_l, \sum_{l=0}^{n} (-2k)^{l-n} q_l\right] = 0,
\end{aligned}$$

where in the last equality we used that

$$W\left[\sum_{l=0}^{n-1} (-2k)^{l+1-n} q_l, \sum_{l=0}^{n} (-2k)^{l-n} q_l\right] = W\left[\sum_{l=0}^{n-1} (-2k)^{l+1-n} q_l, q_n\right].$$

The proofs of (58), (59), and (60) follow the same lines.

6. Rational Solutions of mKdV and Bispectrality

In this section we briefly discuss the bispectral property of the rational solutions of the mKdV hierarchy. This hierarchy is obtained by considering only the odd flows of the AKNS one and by taking $r = q$. We remark that this last condition is preserved by the odd flows of AKNS, but is destroyed by the even ones. Hence the need to restrict to the odd flows. Since we are imposing a severe restriction on the pairs q and r, namely $q = r$, we expect to find more special operators $B(k, \partial_k)$ at least for some choices of $\Theta(x)$. Indeed we shall show in this section that $\Theta(x)$ can be found so that $B(k, \partial_k)$ is similar to a diagonal operator $D(k, \partial_k)$. More specifically,

$$B(k, \partial_k) = SD(k, \partial_k)\, S^{-1},$$

with

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We remark that this matrix S has the property that for $q = r$

$$L \stackrel{\text{def}}{=} S\mathbf{L}S^{-1} = \begin{bmatrix} 0 & \partial_x + q \\ \partial_x - q & 0 \end{bmatrix}, \tag{61}$$

and

$$\operatorname{diag}[H_1, H_2] \stackrel{\text{def}}{=} L^2 = \operatorname{diag}[\partial_x^2 - v_1, \partial_x^2 - v_2], \tag{62}$$

with $v_1 = q^2 + q_x$ and $v_2 = q^2 - q_x$. Note that v_1 and v_2 are the Miura transforms of q. This allows us to relate the bispectral property for q, a rational solution of the mKdV, to its counterpart in the KdV hierarchy.

Theorem 10. *If $r = q$ is a rational solution of the mKdV hierarchy decaying at infinity, then* $\mathbf{L}$ *is bispectral. Moreover, there exists bispectral eigenfunctions $\Psi^{\pm}$ satisfying, for some nonconstant function $\Theta(x)$,*

$$B^{\pm}(k, \partial_k)\, \Psi^{\pm} = \Theta(x)\, \Psi^{\pm},$$

where $B^{\pm} = SD^{\pm}S^{-1}$, with $D^{\pm}$ a nondegenerate diagonal operator independent of x.

Proof. A change of dependent variables $\varphi = S\Psi$ takes the operator $\mathbf{L}$ into L. The fact that q is a rational solution of mKdV decaying at infinity implies that v_1 and v_2 are rational solutions of KdV also decaying at infinity. Now we use some results of [9, Theorems 3.4 and 3.5]. Every pole p of v_1 (resp. v_2) is a regular singular point of $H_1\varphi = 0$ (resp. $H_2\varphi = 0$) and the roots of the indicial equation are $\pm\nu_p$ with $\nu_p \in \mathbb{Z}_{>0}$. We take the eigenfunctions $\varphi_1^{\pm}$ of H_1 that have the asymptotics $\varphi_1^{\pm} = (1 + \mathcal{O}(1/x))\exp(\pm kx)$, as $x \to \infty$. Theorem 3.5 in [9] implies that there exists a polynomial $\Theta(x)$ and $D_1^{\pm}(k, \partial_k)$ such that

$$D_1 \varphi_1^{\pm} = \Theta(x)\, \varphi_1^{\pm}. \tag{63}$$

Furthermore, for Eq. (63) to hold it is necessary and sufficient that the polynomial $\Theta(x)$ satisfy $\Theta^{(2j-1)}(p) = 0$ for every pole p of v_1 and $1 \leqslant j \leqslant \nu_p$. We choose Θ that also has the same property for the poles of v_2. We claim that $\varphi_2^{\pm} \stackrel{\text{def}}{=} k^{-1}(\partial_x - q)\, \varphi_1^{\pm}$ is an eigenfunction of H_2. Indeed, the usual "transference" argument gives

$$H_2 \varphi_2^{\pm} = \frac{1}{k}(\partial_x - q)(\partial_x + q)(\partial_x - q)\, \varphi_1^{\pm} = \frac{1}{k}(\partial_x - q)\, H_1 \varphi_1^{\pm} = k^2 \varphi_2^{\pm},$$

since $\varphi_1^{\pm}$ is an eigenfunction of H_1. Furthermore, the asymptotics of $\varphi_2^{\pm}$ at infinity is the same of $\varphi_1^{\pm}$ times ± 1. Hence, $\varphi_2^{\pm}$ is also a bispectral eigenfunction and there exists $D_2^{\pm}$ such that

$$D_2^{\pm} \varphi_2^{\pm} = \Theta(x)\, \varphi_2^{\pm}, \tag{64}$$

for the polynomial $\Theta(x)$ with the properties above. If we write

$$\varphi^{\pm} = \begin{bmatrix} \varphi_1^{\pm} \\ \varphi_2^{\pm} \end{bmatrix},$$

then

$$L\varphi^{\pm} = k\varphi^{\pm}.$$

Now we put $D = \operatorname{diag}[D_1^{\pm}, D_2^{\pm}]$. Because of Eqs. (63) and (64) this implies that $D^{\pm}\varphi^{\pm} = \Theta(x)\, \varphi^{\pm}$. If, for $l = 1, 2$, we use the asymptotics as $x \to \infty$ of $\varphi_l^{\pm}$ we obtain that the order of $D_l^{\pm}$ is the degree of $\Theta(x)$. This implies the nondegeneracy of D.

We conclude by defining

$$\Psi^{\pm} = S \begin{bmatrix} \varphi_1^{\pm} \\ \varphi_2^{\pm} \end{bmatrix} \tag{65}$$

and by taking $B^{\pm} = SD^{\pm}S^{-1}$. Q.E.D.

In [27] we expand on this subject.

7. Conclusion and Final Remarks

We can summarize the relation between the contribution of this article and the results of [9, 27] using the diagram of Fig. 1.

Let us describe Fig. 1. By rational solutions of the AKNS hierarchy we mean the solutions that can be written as $q=\sigma/\tau$ and $r=\rho/\tau$ with σ, ρ, and τ polynomials satisfying the conditions of Theorem 5. The process, indicated by arrow 2, of specializing to $q=r$ and of considering only the odd time flows leads to the mKdV hierarchy [20]. Arrow 1 indicates that the bispectral property follows as a consequence of Theorem 5. As a corollary of this it follows that if q is in the manifold of rational solutions of the mKdV hierarchy then it also has the bispectral property as long as the gcd condition is satisfied. This is indicated by arrow 4. So far, all we had about the operator $B(k, \partial_k)$ is that it is a nondegenerate matrix differential operator. It turns out that for the rational solutions of the mKdV hierarchy there exists $B^{\pm}$ conjugate to a diagonal operator. As we explained in Theorem 10, arrow 5 indicates this implication. We stress, however, that the diagonal operator in k is obtained only after the change of variables of Eq. (65). Arrow 6 is a consequence of the fact that the Miura transformation maps solutions of the mKdV hierarchy into solutions of the KdV hierarchy. Arrows 7 and 8 are consequences of [9]. We remark that arrow 8 holds as an implication if the space of common solutions of (5) and (6) is at most one dimensional as shown in [9]. Arrow 10

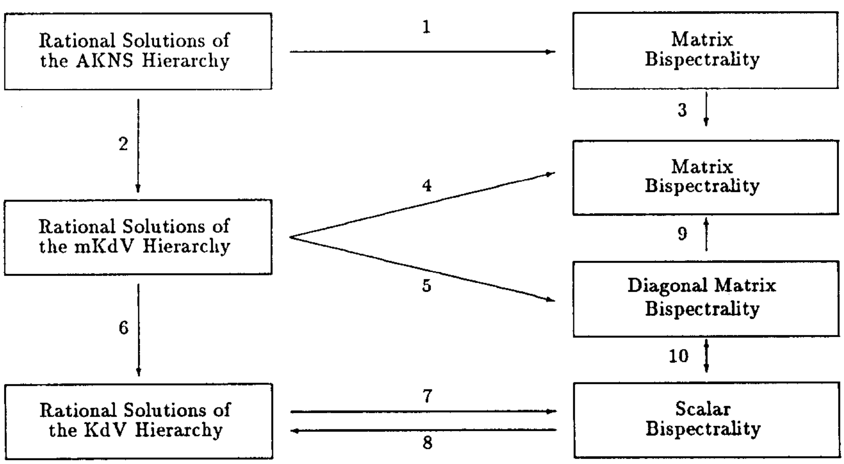

Fig. 1. Diagram describing the relation between the bispectral property and rational solutions of nonlinear evolution equations. The arrowss should be interpreted under the assumptions given in Section 7.

follows from the fact that the components of $\varphi^{\pm} = S\Psi^{\pm}$ are bispectral eigenfunctions for the Schrödinger operators H_1 and H_2 of Eq. (62).

It should be noted that in the present work we did not consider the problem of finding necessary conditions for bispectrality, or the even harder problem of characterizing all the bispectral pairs (q, r). This problem in the matrix case is substantially harder than the one for the Schrödinger operator. We shall address the characterization problem in the future.

We conclude by remarking that the method of constructing differential equations in the spectral parameter for the eigenfunctions associated to the AKNS hierarchy, conceivably, could be adapted to other hierarchies associated to the spectral problem $\partial_x \Psi = (kJ + Q)\Psi$, with more general choices of J and $Q(x)$ than the ones studied here. One example being the case where J is diagonal and $Q(x)$ is off-diagonal. Such hierarchies have been studied in [23]. The crucial part would be to substitute the condition $\gcd(\sigma, \tau) = 1$ or $\gcd(, \rho) = 1$ by some other condition that would ensure the solvability of the linear system that appeared in Proposition 4.

Acknowledgments

I express my gratitude to Professor F. Alberto Grünbaum for introducing me to the subject of this paper and for his helpful supervision of my Ph.D. dissertation, which eventually led to this work. It is also a pleasure to thank Professor Franco Magri for a number of enlightening conversations.

References

1. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, The inverse scattering transform–Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974), 249–315.
2. M. J. Ablowitz and H. Segur, Solitons and the inverse scattering transform, *in* "SIAM Studies in Applied Mathematics," Philadelphia, PA, 1981.
3. M. Adler and J. Moser, On a class of polynomials connected with the KdV equations, *Comm. Math. Phys.* **61** (1978), 1–30.
4. H. Airault, H. McKean, and J. Moser, Rational and elliptic solutions of the KdV equation and a related many body problem, *Comm. Pure Appl. Math.* **30** (1977), 95–148.
5. D. V. Choodnovsky and G. V. Choodnovsky, Pole expansion of nonlinear differential equation, *Nuovo Cimento B* **40** (1977), 339–353.
6. G. Darboux, "Leçons sur la Théorie Générale de Surfaces et les Applications Géométriques du Calcul Infinitésimal, Deuxième Partie," Gauthiers–Villars, Paris, France, 1889.
7. E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Transformation groups for soliton equations, *in* "Proceedings of RIMS Symposium on Non-Linear Integrable Systems" (M. Jimbo and T. Miwa, Eds.), pp. 39–119, World Scientific, Singapore, 1983.
8. P. Deift and E. Trubowitz, Inverse scattering on the line, *Comm. Pure Appl. Math.* **32** (1977), 121–251.
9. J. J. Duistermaat and F. A. Grünbaum, Differential equations in the spectral parameter, *Comm. Math. Phys.* **103** (1986), 177–240.

10. H. Flaschka, A. C. Newell, and T. Ratiu, Kac–Moody Lie algebras and soliton equations II, *Phys. D* **9** (1983), 300–323.
11. C. Gardner, J. Greene, M. D. Kruskal, and R. M. Miura, Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* **19** (1967), 1095–1097.
12. R. Hirota, Direct methods in soliton theory, *in* "Topics in Current Physics" (R. Bullough and P. Caudrey, Eds.), pp. 157–175, Springer-Verlag, New York, 1980.
13. M. Jaulent and I. Miodek, Connection between Zakharov–Shabat and Schrödinger-Type inverse-scattering transforms, *Lett. Nuovo Cimento* **20** (1977), 655–660.
14. M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, II, *Phys. D* **2** (1981), 407–448.
15. M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, III, *Phys. D* **4** (1981), 26–46.
16. M. Jimbo and T. Miwa, Solitons and infinite dimensional lie algebras, *in* "Publications Res. Inst. Math. Sci.," Vol. 19, pp. 943–1001, Kyoto Univ., Kyoto, 1983.
17. M. Jimbo, T. Miwa, and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I, *Phys. D* **2** (1981), 306–352.
18. P. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **31** (1968), 467–490.
19. A. Nakamura and R. Hirota, A new example of explode–decay solitary waves in one-dimension, *J. Phys. Soc. Japan* **54** (1985), 491–499.
20. A. C. Newell, "Solitons in Mathematics and Physics," SIAM, Philadelphia, PA, 1985.
21. S. P. Novikov, S. V. Manakov, V. E. Zakharov, and L. P. Pitaievski, "Theory of Solitons," Plenum, New York, 1984.
22. R. Sachs, On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy, *Phys. D* **30** (1988), 1–27.
23. D. H. Sattinger, Hamiltonian hierarchies on semisimple Lie algebras, *Stud. Appl. Math.* **72** (1985), 65–86.
24. G. Segal and G. Wilson, Loop groups and equations of KdV type, *Inst. Hautes Études Sci. Publ. Math.* **61** (1985), 5–65.
25. G. Wilson, Commuting flows and conservation laws for Lax equations, *Math. Proc. Cambridge Philos. Soc.* **86** (1979), 131–143.
26. V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation waves in nonlinear media, *Soviet Phys. JETP* **34** (1972), 62–89.
27. J. P. Zubelli, "Differential Equations in the Spectral Parameter for Matrix Differential Operators of AKNS Type," Ph.D. thesis, University of California, Berkeley, 1989.
28. J. P. Zubelli, Differential equations in the spectral parameter for matrix differential operators, *Phys. D* **43** (1990), 269–287.